MATH 3063 ABSTRACT ALGEBRA Project 2 Name:
PROF. PAUL BAILEY March 2, 2009

Due Friday, February 6, 2009.

Copy the statement of the problem on a piece of $8\frac{1}{2} \times 11$ piece of blank computer paper, and write the solution underneath. Write neatly. Mathematics should always be written in grammatically correct English, in complete sentences.

For Problems 1 and 2, use the outline provided to complete the proof by filling in the blanks; however, rewrite the entire proof on a separate piece of $8\frac{1}{2} \times 11$ piece of blank computer paper (this will get the style of going about writing the proof "into your fingers").

Definition 1. Let $f : A \to B$.

We say that f is *injective* if
$$f(a_1) = f(a_2) \;\Rightarrow\; a_1 = a_2.$$

We say that f is *surjective* if
$$\forall b \in B\, \exists a \in A \text{ such that } f(a) = b.$$

We say that f is *bijective* if f is injective and surjective.

Problem 1. Let $f : A \to B$ and $g : B \to C$. Suppose that f is surjective and $g \circ f$ is injective. Show that g is injective.

Proof. Let $b_1, b_2 \in B$ such that $g(b_1) = g(b_2)$. We wish to show that $b_1 = b_2$.

Since f is surjective, there exist $a_1, a_2 \in$ ______________ such that

$f(a_1) =$ ______________ and $f(a_2) =$ ______________.

Applying g to these equations gives $g(f(a_1)) =$ ______________ and $g(f(a_2)) =$ ______________.

But $g(b_1) = g(b_2)$, and since $g \circ f$ is injective, $a_1 =$ ______________.

Thus $f(a_1) =$ ______________, that is, $b_1 = b_2$.

Therefore g is injective. □

Problem 2. Let $f : A \to B$ and $g : B \to C$. Suppose that g is injective and $g \circ f$ is surjective. Show that f is surjective.

Proof. Let $b \in B$. We wish to find $a \in A$ such that $f(a) = b$.

Let $c = g($______________$)$.

Since $g \circ f$ is surjective, there exists $a \in A$ such that ______________ $= c$,

that is, $g(f(a)) = g(b)$.

Since g is injective, ______________ $= b$.

Therefore f is surjective. □

Problem 3. Consider the relationship between composition and bijectivity.

(a) Show that the composition of injective functions is injective.

(b) Show that the composition of surjective functions is surjective.

(c) Conclude that the composition of bijective functions is bijective.

Definition 2. Let $f : A \to B$ and $g : B \to A$.
We say that g is an *inverse* of f if $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.
We say that g is a *left inverse* of f if $g \circ f = \mathrm{id}_A$.
We say that g is a *right inverse* of f if $f \circ g = \mathrm{id}_B$.
We say that f is *invertible* if the exists an inverse for f.
We say that f is *left invertible* if the exists a left inverse for f.
We say that f is *right invertible* if the exists a right inverse for f.

Problem 4. Consider the existence of left and right inverses by giving examples.

(a) Give an example of a function which is left invertible but not invertible.

(b) Give an example of a function which is right invertible but not invertible.

Problem 5. Consider the relationship between invertibility and bijectivity.

(a) Show that a function is left invertible if and only if it is injective.

(b) Show that a function is right invertible if and only if it is surjective.

(c) Conclude that a function is invertible if and only if it is bijective.

Definition 3. Let $P, Q \in \mathbb{R}^2$ be given by $P = (x_1, y_1)$ and $Q = (x_1, y_2)$. The *distance* from P to Q is

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

An *isometry* of $\mathbb{R}^2$ is a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $d(f(P), f(Q)) = d(P, Q)$.
Three types of isometries are translations, rotations, and reflections.
A translation is described by (h, k), where $(x, y) \mapsto (x + h, y + k)$.
A rotation is described by (a, b, θ), where (a, b) is a fixed point and θ is the angle of rotation.

Problem 6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry such that $f = T \circ R$, where T is the translation described by (h, k) and R is the rotation described by (a, b, θ). Suppose $f(5, 0) = (1, 2)$ and $f(7, 0) = (2, 2 + \sqrt{3})$. Find h, k, a, b, θ.